

TAIL ASYMPTOTICS AND ESTIMATION FOR ELLIPTICAL DISTRIBUTIONS

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ABSTRACT. Let (X, Y) be a bivariate elliptical random vector with associated random radius in the Gumbel max-domain of attraction. In this paper we obtain a second order asymptotic expansion of the joint survival probability $\mathbf{P}\{X > x, Y > y\}$ for x, y large. Further, based on the asymptotic bounds we discuss some aspects of the statistical modelling of joint survival probabilities and the survival conditional excess probability.

1. INTRODUCTION

Let (S_1, S_2) be a rotational invariant (spherical) bivariate random vector with associated random radius $R := \sqrt{S_1^2 + S_2^2}$. The basic distributional properties of spherical random vectors are obtained in Cambanis et al. (1981). So if $R > 0$ almost surely, then we have the stochastic representation

$$(S_1, S_2) \stackrel{d}{=} (RU_1, RU_2),$$

where the bivariate random vector (U_1, U_2) is independent of the associated random radius R and uniformly distributed on the unit circle of \mathbb{R}^2 ($\stackrel{d}{=}$ stands for equality of distribution functions). Linear combinations of spherical random vectors define a larger class of random vectors, namely that of elliptical random vectors. Canonical examples are the Gaussian and Kotz distributions (see Fang et al. (1990), Kotz et al. (2000), or Reiss and Thomas (2007)).

In this paper we consider a bivariate elliptical random vector defined in terms of (S_1, S_2) and the pseudo-correlation coefficient $\rho \in (-1, 1)$ via the stochastic representation

$$(1) \quad (X, Y) \stackrel{d}{=} (S_1, \rho S_1 + \sqrt{1 - \rho^2} S_2).$$

Since for any a, b two constants (see Lemma 6.2 of Berman (1982))

$$(2) \quad aS_1 + bS_2 \stackrel{d}{=} S_1 \sqrt{a^2 + b^2}$$

we have $X \stackrel{d}{=} Y \stackrel{d}{=} S_1$. Referring to Cambanis et al. (1981) the random variable S_1 is symmetric about 0, and furthermore $S_1^2 \stackrel{d}{=} R^2 W$, with W a Beta random variable with parameters $1/2, 1/2$ independent of R , implying that the distribution function of X and Y is completely known if the distribution function F of R is specified.

The basic distributional properties of elliptical random vectors are well-known, see e.g., Kotz (1975), Cambanis et al. (1981), Anderson and Fang (1990), Fang et. al (1990), Fang and Zhang (1990), Berman (1992), Gupta and Varga (1993), Kano (1994), Szabłowski (1998), or Kotz et al. (2000) among several others.

The tail asymptotic behaviour of each component, say of X , can be determined under some assumptions on the tail asymptotics of R . The main work in this direction is done in Carnal (1970), Gale (1980), Eddy and Gale (1981), Berman (1982, 1983, 1992) among several others. For instance Berman (1992) shows the exact asymptotic behaviour of S_1 if R has distribution function in the Gumbel max-domain of attraction. With motivation from Berman's work, Hashorva (2007) obtains an exact asymptotic expansion of the bivariate survival probability

$$\mathbf{P}\{X > x, Y > ax\}, \quad a \leq 1$$

letting x tend to ∞ . See also Asimit and Jones (2007) for a partial result.

The main impetus for the present article comes from the recent deep contribution Abdous et al. (2007). We derive in this paper a refinement of the asymptotic expansion of the joint survival probability obtained in Hashorva (2007).

Date: May 15, 2008.

2000 Mathematics Subject Classification. Primary 60F05; Secondary 60G70.

Key words and phrases. Elliptical random vectors; exact asymptotics; second order bounds; Gumbel max-domain of attraction; conditional distribution; conditional quantile function; joint survival probability; statistical estimation.

This is achieved assuming some second order asymptotic bounds on the tail asymptotics of the distribution function F as suggested in Abdous et al. (2007).

Our results are of a certain theoretical interest providing detailed asymptotic expansions for a classical problem of probability theory - tail asymptotics of random vectors. Further, based on our novel results, we suggest statistical estimators of the joint survival probability, conditional distribution and related quantile function.

We choose the following order for the rest of the paper: In Section 2 we present the main results. Illustrating examples follow in Section 3. In Section 4 and Section 5 we discuss some statistical aspects concerning the estimation of the joint survival probability, conditional survival and conditional quantile functions. Proofs and related asymptotics are relegated to Section 6.

2. ASYMPTOTIC BOUNDS

Let (X, Y) be a bivariate elliptical random vector as in (1), where the associated random radius R has distribution function F with upper endpoint $x_F \in (0, \infty]$ and $F(0) = 0$. Hashorva (2007) derives an asymptotic expansion of the tail probability $\mathbf{P}\{X > x, Y > ax\}$, $a \in (-\infty, 1]$ for $x \uparrow x_F$ assuming that F is in the Gumbel max-domain of attraction, i.e.,

$$(3) \quad \lim_{u \uparrow x_F} \frac{1 - F(u + s/w(u))}{1 - F(u)} = \exp(-s), \quad \forall s \in \mathbb{R},$$

with w some positive scaling function. Refer to Galambos (1987), Resnick (1987), Reiss (1989), Embrechts et al. (1997), Falk et al. (2004), Kotz and Nadarajah (2005), or de Haan and Ferreira (2006) for details on the Gumbel max-domain of attraction.

It is well-known (see e.g., Resnick (1987)) that (3) is equivalent with the fact that the distribution function F has the following representation

$$(4) \quad 1 - F(x) = d(x)[1 - F^*(x)], \quad x < x_F,$$

with $d(x)$ a positive function converging to $d \in (0, \infty)$ as $x \uparrow x_F$ and

$$(5) \quad F^*(x) = 1 - \exp\left(-\int_{z_0}^x w(s) ds\right), \quad x < x_F,$$

where z_0 is a finite constant in the left neighbourhood of the upper endpoint x_F . The distribution F^* is referred to in the sequel as the von Mises distribution related to F , whereas w as the von Mises scaling function of F .

Under the Gumbel max-domain of attraction assumption on F (see Hashorva (2007)) we have

$$(6) \quad \mathbf{P}\{X > x, Y > ax\} = \frac{(1 + o(1))c_1}{\sqrt{xw(\alpha_{\rho,x,ax}x)}} \mathbf{P}\{X > \alpha_{\rho,x,ax}x\}, \quad x \rightarrow \infty,$$

with c_1 a known constant, provided that $a \in (\rho, 1]$ and $x_F = \infty$, where $\alpha_{\rho,x,y}$ is defined by

$$(7) \quad \alpha_{\rho,x,y} := \sqrt{1 + (y/x - \rho)^2/(1 - \rho^2)}.$$

For x, y positive such that $y/x \geq a > \rho$ we have $\alpha_{\rho,x,y} > 1$. Hence, since the scaling function w satisfies

$$(8) \quad \lim_{u \uparrow x_F} uw(u) = \infty, \quad \text{and} \quad \lim_{u \uparrow x_F} w(u)(x_F - u) = \infty, \quad \text{if } x_F < \infty$$

we conclude that the joint tail asymptotics in (6) is faster than the componentwise tail asymptotics.

It turns out that for $y < \rho x$ or y close enough to ρx the tail asymptotics of interest is up to a constant the same as that of $\mathbf{P}\{X > x\}$. For the bivariate Gaussian distribution it is well-known that this fact is closely related to the so called Savage condition, see Dai and Mukherjea (2001), Hashorva and Hüsler (2003), or Hashorva (2005a) for details. For elliptical distributions the case where y is close to ρx has been considered independently in Gale (1980), Eddy and Gale (1981) and Berman (1982, 1983, 1992).

In this paper we are interested in a refinement of (6) which will be achieved under extra costs related to a second order assumption on F .

Explicitly, as suggested in Abdous et al. (2007) we impose the following assumption:

A1. Suppose that the distribution function F satisfies (4) with von Mises distribution function F^* , and assume further that there exist positive measurable functions $A_i, B_i, i = 1, 2$ such that for a scaling function w for which (3) holds we have

$$(9) \quad \left| \frac{1 - F^*(u + x/w(u))}{1 - F^*(u)} - \exp(-x) \right| \leq A_1(u)B_1(x)$$

and

$$(10) \quad \left| d(u + x/w(u)) - d(u) \right| \leq A_2(u)B_2(x)$$

for all u large and any $x \geq 0$. Furthermore, we assume that $\lim_{u \rightarrow \infty} A_1(u) = \lim_{u \rightarrow \infty} A_2(u) = 0$, and B_1, B_2 are locally bounded on any compact interval of $[0, \infty)$.

Set in the following (whenever Assumption A1 is assumed)

$$(11) \quad A(x) := A_1(x) + A_2(x), \quad B(x) := B_1(x) + \exp(-x)B_2(x), \quad \forall x > 0.$$

We note in passing that $A_2(x) = B_2(x) = 0, x > 0$ is the original condition in the aforementioned paper, where it is shown (see Lemma 7 therein) that the class of distribution functions satisfying Assumption A1 is quite large.

We consider in the sequel only distribution functions F with an infinite upper endpoint. Further we assume that $\rho \in [0, 1)$. We state now the main result of this paper:

Theorem 1. Let (S_1, S_2) be a bivariate spherical random vector with associate random radius $R \sim F$, where the distribution function F has an infinite upper endpoint and $F(0) = 0$. Let (X, Y) be a bivariate elliptical random vector with stochastic representation $(X, Y) \stackrel{d}{=} (S_1, \rho S_1 + \sqrt{1 - \rho^2} S_2), \rho \in [0, 1)$. Assume that (3) holds with the scaling function w and Assumption A1 is valid with $A_i, B_i, i = 1, 2$ positive measurable functions (with A, B as in (11)).

a) If x, y are positive constants such that

$$(12) \quad y \in (\rho x, x], \quad \alpha_{\rho, x, y} \geq c > 1,$$

and $\int_0^\infty B(s) ds < \infty$, then for all x large

$$(13) \quad \mathbf{P}\{X > x, Y > y\} = \frac{\alpha_{\rho, x, y} K_{\rho, x, y}}{2\pi} \frac{1 - F(\alpha_{\rho, x, y} x)}{xw(\alpha_{\rho, x, y} x)} \left[1 + O\left(A(\alpha_{\rho, x, y} x) + \frac{1}{xw(\alpha_{\rho, x, y} x)} \right) \right],$$

with $\alpha_{\rho, x, y}$ as defined in (7) and $K_{\rho, x, y}$ given by

$$(14) \quad K_{\rho, x, y} := \frac{x^2(1 - \rho^2)^{3/2}}{(x - \rho y)(y - \rho x)} \in (0, \infty).$$

b) Set $h(x) := xw(x), x > 0$, and let $z_x, x \in \mathbb{R}$ be given constants such that $|z_x| < K < \infty$ for all x large. If further $\int_0^\infty s^{-1/2} B(s) ds < \infty$, then for all x large and $y := x[\rho + z_x \sqrt{1 - \rho^2}/\sqrt{h(x)} + \rho/h(x)]$ we have

$$(15) \quad \mathbf{P}\{X > x, Y > y\} = \frac{1}{\sqrt{2\pi}} \frac{1 - F(x)}{\sqrt{h(x)}} [1 - \Phi(z_x)] \left[1 + O\left(A(x) + \frac{1}{h(x)} \right) \right],$$

where Φ denotes the standard Gaussian distribution on \mathbb{R} .

c) If x, y are positive constants with $y < ax, a \in (0, \rho), \rho > 0$ and $\int_0^\infty s^{-1/2} B(s) ds < \infty$, then

$$(16) \quad \mathbf{P}\{X > x, Y > y\} = \frac{1}{\sqrt{2\pi}} \frac{1 - F(x)}{\sqrt{h(x)}} \left[1 + O\left(A(x) + \frac{1}{\sqrt{h(x)}} \left[\frac{1}{\sqrt{h(x)}} + \frac{1 - F(\alpha_{\rho, x, y} x)}{1 - F(x)} \right] \right) \right]$$

is valid for all large x .

Remark 1. a) By the assumption on F (recall (8)) we have $\lim_{x \rightarrow \infty} (xw(x))^{-1} = 0$. If we assume that $\lim_{u \rightarrow \infty} A(u) = 0$ holds, then we have

$$O\left(A(\alpha_{\rho, x, y} x) + \frac{1}{xw(\alpha_{\rho, x, y} x)} \right) = o(1), \quad x \rightarrow \infty.$$

Consequently by statement a) in Theorem 1

$$\mathbf{P}\{X > x, Y > y\} = (1 + o(1)) \frac{\alpha_{\rho, x, y} K_{\rho, x, y}}{2\pi} \frac{1 - F(\alpha_{\rho, x, y} x)}{xw(\alpha_{\rho, x, y} x)}, \quad x \rightarrow \infty,$$

and if $y = ax(1 + o(1))$, $a \in (\rho, 1]$, then for all large x we have

$$\mathbf{P}\{X > x, Y > y\} = (1 + o(1)) \frac{\alpha_{\rho,a} K_{\rho,a}}{2\pi} \frac{1 - F(\alpha_{\rho,a}x)}{xw(\alpha_{\rho,a}x)}, \quad x \rightarrow \infty,$$

holds with

$$(17) \quad \alpha_{a,\rho} := \sqrt{(1 - 2a\rho + a^2)/(1 - \rho^2)} > 1, \quad K_{\rho,a} := \frac{(1 - \rho^2)^{3/2}}{(1 - a\rho)(a - \rho)} \in (0, \infty).$$

b) Sufficient conditions for (9) to hold are derived in Lemma 7 of Abdous et al. (2007). An instance is when $\lim_{x \rightarrow \infty} w(tx)/w(x) = t^{\theta-1}$, $\theta > 0$, $\forall t > 0$, i.e., the scaling function w defining F^* is regularly varying at infinity with index $\theta - 1$. Under the assumptions of Lemma 7 in the aforementioned paper we may choose

$$A_1(u) = O(|(1/w(u))'|), \quad B_1(x) = (1 + x)^{-\kappa}, \quad u > 0, x > 0, \kappa > 1.$$

c) We have $\alpha_{\rho,x,y}^2 = 1$ iff $y = \rho x$ and for any $y \in (0, x]$, $x > 0$

$$1 \leq \alpha_{\rho,x,y}^2 \leq \frac{2}{1 + \rho}, \quad \rho \in (-1, 1).$$

d) Since (3) implies that $1 - F$ is rapidly varying (see e.g., Resnick (1987) or de Haan and Ferreira (2006)) and $\alpha_{\rho,x,y} \geq C > 1$, then we have

$$\lim_{x \rightarrow \infty} \frac{1 - F(\alpha_{\rho,x,y}x)}{1 - F(x)} = 0,$$

consequently in (16) we have

$$O\left(A(x) + \frac{1}{\sqrt{xw(x)}} \left[\frac{1}{\sqrt{xw(x)}} + \frac{1 - F(\alpha_{\rho,x,y}x)}{1 - F(x)} \right]\right) = O\left(A(x) + \frac{o(1)}{\sqrt{xw(x)}}\right).$$

We give next an alternative expansion of the tail probability under consideration assuming $y \in (\rho x, x]$.

Theorem 2. *Let $(X, Y), \rho, F$ be as in Theorem 1. Suppose that (3) holds with the scaling function w and further Assumption A1 is satisfied with A_i, B_i positive measurable functions. Let x, y be two positive constants such that (12) holds. If $\int_0^\infty \max(1, 1/\sqrt{s}) B(s) ds < \infty$ holds with B as defined in (11), then for all x large we have*

$$(18) \quad \mathbf{P}\{X > x, Y > y\} = \frac{\alpha_{\rho,x,y}^{3/2} K_{\rho,x,y}}{\sqrt{2\pi xw(\alpha_{\rho,x,y}x)}} \mathbf{P}\{X > \alpha_{\rho,x,y}x\} \left[1 + O\left(A(\alpha_{\rho,x,y}x) + \frac{1}{xw(\alpha_{\rho,x,y}x)}\right) \right],$$

with $\alpha_{\rho,x,y}, K_{\rho,x,y}$ as defined in (7) and (14), respectively.

Next, we consider the implications of our results for the tail asymptotics of the conditional survival probability $\mathbf{P}\{Y > y|X > x\}$.

Theorem 3. *Under the assumptions and notation of Theorem 2 we have for all x, y large*

$$(19) \quad \mathbf{P}\{Y > y|X > x\} = \frac{\alpha_{\rho,x,y}^{3/2} K_{\rho,x,y}}{\sqrt{2\pi}} g_1(\alpha_{\rho,x,y}, x) = \frac{\alpha_{\rho,x,y}^{3/2} K_{\rho,x,y}}{\sqrt{2\pi}} g_2(\alpha_{\rho,x,y}, x),$$

where

$$(20) \quad g_1(\alpha_{\rho,x,y}, x) := \frac{\sqrt{w(x)/(\alpha_{\rho,x,y}x)}}{w(\alpha_{\rho,x,y}x)} \frac{1 - F(\alpha_{\rho,x,y}x)}{1 - F(x)} \left[1 + O\left(A(\alpha_{\rho,x,y}x) + \frac{1}{xw(\alpha_{\rho,x,y}x)}\right) \right],$$

$$(21) \quad g_2(\alpha_{\rho,x,y}, x) := \frac{\mathbf{P}\{X > \alpha_{\rho,x,y}x\}}{\sqrt{xw(\alpha_{\rho,x,y}x)} \mathbf{P}\{X > x\}} \left[1 + O\left(A(\alpha_{\rho,x,y}x) + \frac{1}{xw(\alpha_{\rho,x,y}x)}\right) \right].$$

Furthermore, we have

$$(22) \quad \lim_{x \rightarrow \infty} \mathbf{P}\{Y > y|X > x\} = 0,$$

and for any $a \in (\rho, 1]$, $z \geq 0$

$$(23) \quad \mathbf{P}\{Y > ax + z/w(\alpha_{a,\rho}x)|X > x\} = \frac{\alpha_{a,\rho}^{3/2} K_{a,\rho}}{\sqrt{2\pi}} \exp(-\lambda_{a,\rho}z) g_1(\alpha_{a,\rho}, x)$$

$$(24) \quad = \frac{\alpha_{a,\rho}^{3/2} K_{a,\rho}}{\sqrt{2\pi}} \exp(-\lambda_{a,\rho} z) g_2(\alpha_{a,\rho}, x)$$

hold locally uniformly with respect to z , with $\alpha_{a,\rho}, K_{a,\rho}$ as defined in (17) and

$$(25) \quad \lambda_{\rho,a} := \frac{a - \rho}{\sqrt{(1 - \rho^2)(1 - 2a\rho + a^2)}} > 0.$$

Convergence to a positive constant in (22) is achieved when y depends on x being close to ρx for x large. See Gale (1980), Berman (1982, 1983, 1992), Abdous et al. (2007), Hashorva (2006, 2007) for elliptical case, and Hashorva et al. (2007) for the more general case of Dirichlet distributions.

Remark 2. a) A slightly more general setup is when Assumption A1 is reformulated considering positive measurable functions $A_i, B_i, i \geq 1$ such that for all $x > 0$ and u large

$$\left| \frac{1 - F^*(u + x/w(u))}{1 - F^*(u)} - \exp(-x) \right| \leq \sum_{i=1}^{\infty} A_i(u) B_i(x) =: \Psi(u, x)$$

is valid with $\lim_{u \rightarrow \infty} \sup_{i \geq 1} A_i(u) = 0, i \geq 1$, $\Psi(u, x)$ finite for all $x > 0$ and u large.

b) In the asymptotic results above the scaling function w appears prominently. One choice for the scaling function is the von Mises scaling function, i.e., w defines the von Mises distribution function F^* in (5). We can choose however another scaling function \bar{w} defined asymptotically by

$$(26) \quad \bar{w}(u) := \frac{(1 + o(1))[1 - F(u)]}{\int_u^{x_F} [1 - F(s)] ds}, \quad u \uparrow x_F.$$

Note that the bounding functions $A_i, B_i, i = 1, 2$ in the Assumption A1 depend on which concrete scaling function we choose. In view of Lemma 16 in Abdous et al. (2007) (see its assumptions and Lemma 7 therein) if w_F is the von Mises scaling function defining F^* in (5) and \bar{w} is another scaling function defined by (26), then (9) holds with \bar{w} instead of w_F , and \bar{A}_1 instead of A_1 , where \bar{A}_1 is defined by

$$(27) \quad \bar{A}_1(u) = O\left(|(1/w_F(u))'| + |w_F(u) - \bar{w}(u)|/\bar{w}(u)\right), \quad \forall u > 0.$$

If F is a von Mises distribution function, then we can make use of Lemma 7 in Abdous et al. (2007) (provided the assumptions of that lemma hold).

We discuss in the next lemma the case F is a mixture distribution.

Lemma 4. Let $F_i, i \geq 1$ be von Mises distribution functions with the same scaling function w and upper endpoint infinity. Suppose that the Assumption A1 is satisfied for all $F_i, i \geq 1$ with corresponding functions $w, A_{i1}, B_{i1}, i \geq 1$, and $A_{i2}, B_{i2}, i \geq 1$ identical to 0. If F is another distribution function defined by

$$F(x) = \sum_{i=1}^{\infty} a_i F_i(x), \quad \text{with } a_i > 0, i \geq 1 : \sum_{i=1}^{\infty} a_i = 1$$

for all x large, then F is in the Gumbel max-domain of attraction. If F is the distribution function of the random radius R in Theorem 2 and Theorem 3, then both these theorems hold with

$$A(u) = \sum_{i=1}^{\infty} A_{i1}(u), \quad u > 0,$$

provided that $\sum_{i=1}^{\infty} a_i \int_0^{\infty} B_{i1}(s) ds < \infty$ and $A(u)$ is bounded for all large u with $\lim_{u \rightarrow \infty} A(u) = 0$.

3. EXAMPLES

We present next three illustrating examples.

Example 1. [Kotz Type I]

Let $(X, Y) = R(U_1, \rho U_1 + \sqrt{1 - \rho^2} U_2)$, $\rho \in [0, 1]$ with R a positive random radius independent of the bivariate random vector (U_1, U_2) which is uniformly distributed on the unit circle of \mathbb{R}^2 . We call (X, Y) a Kotz Type I elliptical random vector if

$$1 - F(x) = Cx^N \exp(-cx^{\delta}), \quad c > 0, C > 0, \delta > 0, N \in \mathbb{R}$$

for any $x > 0$. Set $w(u) := c\delta u^{\delta-1}$, $u > 0$. For any $x \in \mathbb{R}$ we obtain

$$\lim_{u \rightarrow \infty} \frac{\mathbf{P}\{R > u + x/w(u)\}}{\mathbf{P}\{R > u\}} = \exp(-x).$$

Consequently, F is in the Gumbel max-domain of attraction with the scaling function w . Further, we have

$$\mathbf{P}\{X > u\} = (1 + o(1)) \frac{C}{\sqrt{2r\delta\pi}} u^{N-\delta/2} \exp(-ru^\delta), \quad u \rightarrow \infty.$$

The von Mises scaling function w is given by

$$w(u) = c\delta u^{\delta-1} - Nu^\eta, \quad \eta := -1.$$

Hence (27) implies that the approximation under consideration holds choosing $A(u) := A_1(u)$, $u > 0$, where $A_1(u) := O(u^{-\delta})$, $u > 0$. Further we can take $B_1(x) = (1+x)^{-\kappa}$, $\kappa > 1$ and $B_2(x) = 0$, $x > 0$. Hence the second order assumption in our results above is satisfied. We may thus write for x large and $y \in (\rho x, x]$ such that $\alpha_{\rho,x,y} > c > 1$

$$\begin{aligned} \mathbf{P}\{X > x, Y > y\} &= (1 + o(1)) \frac{C\alpha_{\rho,x,y}^{2-\delta+N} K_{\rho,x,y} x^{N-\delta}}{2r\delta\pi} \exp(-r(\alpha_{\rho,x,y} x)^\delta) [1 + O(x^{-\delta})] \\ &= (1 + o(1)) \frac{\alpha_{\rho,x,y}^{2-\delta/2} K_{\rho,x,y}}{\sqrt{2r\delta\pi}} x^{-\delta/2} \frac{C\alpha_{\rho,x,y}^{N-\delta/2} K_{\rho,x,y} x^{N-\delta}}{\sqrt{2r\delta\pi}} \exp(-r(\alpha_{\rho,x,y} x)^\delta) [1 + O(x^{-\delta})] \\ &= (1 + o(1)) \frac{\alpha_{\rho,x,y}^{2-\delta/2} K_{\rho,x,y}}{\sqrt{2r\delta\pi}} x^{-\delta/2} \mathbf{P}\{X > \alpha_{\rho,x,y} x\} [1 + O(x^{-\delta})] \\ &= (1 + o(1)) \frac{\alpha_{\rho,x,y}^{2-\delta/2} K_{\rho,x,y}}{\sqrt{2r\delta\pi}} \mathbf{P}\{X^* > \alpha_{\rho,x,y} x\} [1 + O(x^{-\delta})], \end{aligned}$$

with (X^*, Y^*) another Kotz Type I random vector with coefficients $C, N^* = N - \delta/2, r, \delta$.

Example 2. [Tail Equivalent Distributions]

Let G be a von Mises distribution function with scaling function w such that (9) holds with functions A_1, B_1 , and let $F_{\gamma,\tau}, \gamma, \tau > 0$ be another distribution function with infinite upper endpoint. Assume further that

$$(28) \quad 1 - F_{\gamma,\tau}(x) = (1 + ax^{-\gamma} + O(x^{-\tau\gamma})) [1 - G(x)], \quad \forall x > 0,$$

with γ, τ two positive constants. Clearly, F and G are tail equivalent since

$$\lim_{x \rightarrow \infty} \frac{1 - F_{\gamma,\tau}(x)}{1 - G(x)} = 1.$$

Suppose further that $w(x) := c\delta x^{\delta-1}$, $c > 0, \delta > 0$, and set $d(x) := (1 + ax^{-\gamma} + O(x^{-\tau-\gamma})), x > 0$. We have for all large u and $x > 0$

$$\begin{aligned} &|d(u + x/w(u)) - d(u)| \\ &= au^{-\gamma} \left[(1 + x/(uw(u)))^{-\gamma} - 1 \right] - u^{-\tau-\gamma} \left[1 + O((1 + x/(uw(u)))^{-\tau\gamma}) \right] \\ &= A_2(u)B_2(x), \end{aligned}$$

where $A_2(u) := O(u^{-[\gamma+\min(\tau,\delta)]})$, $u > 0$ and $B_2(x)$ is such that $\int_0^\infty B_2(x) \exp(-x) dx < \infty$. Hence our asymptotic results in the above theorems hold for such F with the function A defined by $A(u) := A_1(u) + A_2(u)$, $u > 0$.

Example 3. [Regularly Varying w]

Consider F a von Mises distribution function in the Gumbel max-domain of attraction with the scaling function $w(x) = F'(x)/[1 - F(x)]$, $x > 0$ defined by

$$w(x) := \frac{c\delta x^{\delta-1}}{1 + t_1(x)} = c\delta x^{\delta-1}(1 + o(1)), \quad c > 0, \delta > 0, \quad \forall x > 0,$$

which implies (Abdous et al. (2007)) that all x large

$$\mathbf{P}\{R > x\} = \exp(-cx^\delta(1 + t_2(x)))$$

holds, where $t_i(x)$, $i = 1, 2$ are two regularly varying functions with index $\eta\delta, \eta < 0$.

Now choosing $\bar{w}(x) := c\delta x^{\delta-1}$ instead of $w(x)$ we conclude that the second order correction function A_1 satisfies

$$A_1(u) := O(u^{-\delta} + u^{\eta\delta} L_1(u)), \quad u > 0,$$

with $L_1(u)$ a positive slowly varying function, i.e., $\lim_{u \rightarrow \infty} L_1(tu)/L_1(u) = 1, t > 0$. It can be easily checked that our main theorems above hold for this case with $A(u) := A_1(u), u > 0$.

4. ESTIMATION OF JOINT SURVIVAL PROBABILITY

Consider the estimation of the joint survival probability $\mathbf{P}\{X > x, Y > y\}$ for x, y large, with (X, Y) a bivariate random vector satisfying the assumptions of Theorem 2. We discuss first the implications of (13), (18), and then outline the estimation motivated by (15).

Estimation based on (13), (18)

Consider the case $x = y$ is large. The constants $\alpha_{\rho,x,x}, K_{\rho,x,x}$ do not depend on x and y (given in (17) for $a = 1, \rho \in [0, 1]$). Both asymptotic expansions can be written as

$$\mathbf{P}\{X > x, Y > x\} = q_1(x, \rho, w, F) = q_2(x, \rho, w, G_\zeta),$$

with F and G_ζ the distribution function of R and $Z_{\zeta,\rho}$, respectively.

If $(X_i, Y_i), 1 \leq i \leq n$ are independent copies of (X, Y) , then a \sqrt{n} -consistent estimator of ρ is available from the literature. The more difficult part is the estimation of the tails and the function w .

If we restrict ourselves to distribution functions F in the Gumbel max-domain of attraction with von Mises scaling function

$$w(x) = c\delta x^{\delta-1}, \quad c > 0, \delta > 0, \quad x > 0,$$

then w can be estimated utilising the techniques in Abdous et al. (2007), where estimators for c and δ are constructed using previous results of Girard (2004), Gardes and Girard (2006). See also the recent contribution Diebolt et al. (2007).

The estimation of the tail $1 - F$ can be performed for instance if we restrict ourself to the case of Example 3 above. Advanced extreme value statistics provide estimation of Gumbel tail and related quantiles under second order assumptions. See the recent deep monographs de Haan and Ferreira (2006), Reiss and Thomas (2007). If we use (18) then second order asymptotic condition on the distribution of X need to be imposed.

Note in passing that the scaling function w appears in the assumption on F , which on the turn implies that both X and Y have distribution functions in the Gumbel max-domain of attraction with the same scaling function w . Consequently we may estimate w alternatively utilising only the observations $X_i, 1 \leq i \leq n$, or more generally we may estimate w from the observations

$$Z_{\zeta,\rho,i} = \frac{\zeta X_i + Y_i}{\sqrt{\zeta^2 + 2\zeta\rho + 1}} \stackrel{d}{=} X, \quad \zeta \in \mathbb{R}, \quad i = 1, \dots, n.$$

Further, instead of estimating the tail $1 - F$ we may estimate the tail $1 - G_\zeta$. If we use the random points $X_i, 1 \leq i \leq n$ to estimate w the advantage is that the estimator of ρ is not involved. The disadvantage is that the second order correction is a consequence of an assumption on R and not on X .

Estimation based on (15)

If x, y are large positive constants then making use of the approximation in (15) we may write (set $h(x) := \sqrt{xw(x)}, x > 0$)

$$\mathbf{P}\{X > x, Y > y\} \approx \frac{1}{\sqrt{2\pi}} \frac{1 - F(x)}{\sqrt{h(x)}} \left[1 - \Phi \left((y/x - \rho - \rho/h(x)) \sqrt{h(x)/(1 - \rho^2)} \right) \right],$$

hence an estimator of the probability of interest can be constructing by the right hand side of the above approximation. Again we have the same estimation issues for the tail $1 - F$ and $w(x)$ where x is large as above.

5. ESTIMATION OF CONDITIONAL SURVIVAL AND QUANTILE FUNCTION

Let $(X, Y), (X_i, Y_i), 1 \leq i \leq n$, be as in the previous section. Our asymptotic results above can be employed for the estimation of the conditional excess survival function

$$\Psi(y, x) := \mathbf{P}\{Y > y | X > x\},$$

where x, y are large. The estimation of the conditional distribution $1 - \Psi(y, x)$ is discussed in Abodus et al. (2007). Clearly, one way to address this problem is to consider the estimation of the joint and marginal survival probabilities $\mathbf{P}\{X > x, Y > y\}, \mathbf{P}\{X > x\}$, separately. As noted in the aforementioned paper for large values of x the empirical distribution function is useless since no observations might fall in the relevant regions.

Our suggestion for the estimation of $\Psi(y, x)$ is motivated by the asymptotic relations shown in Theorem 3. Under the assumptions of that theorem for all large x and any $z \in \mathbb{R}$ we have

$$\begin{aligned}\Psi(x + z/w(\alpha_{1,\rho}x), x) &= \frac{\alpha_{1,\rho}^{3/2} K_{1,\rho}}{\sqrt{2\pi}} \exp(-\lambda_{1,\rho}z) g_1(\alpha_{1,\rho}, x) \\ &= \frac{\alpha_{1,\rho}^{3/2} K_{1,\rho}}{\sqrt{2\pi}} \exp(-\lambda_{1,\rho}z) g_2(\alpha_{1,\rho}, x),\end{aligned}$$

with $\alpha_{1,\rho}, K_{1,\rho}$ and $\lambda_{1,\rho}$ as in (17) and (25), respectively.

Let $\hat{w}_n, \hat{g}_{n1}, \hat{g}_{n2}, \hat{\rho}_n$ be estimators of the function w, g_1, g_2 and ρ , respectively, and set

$$\hat{\alpha}_n := \alpha_{1,\hat{\rho}_n}, \quad \hat{\lambda}_n := \lambda_{1,\hat{\rho}_n}, \quad n \geq 1.$$

Then we may estimate $\Psi(y, x)$ by

$$\begin{aligned}\hat{\Psi}_{n,1}(y, x) &:= \frac{\hat{\alpha}_n^{3/2} K_{1,\hat{\rho}_n}}{\sqrt{2\pi}} \exp(-\hat{\lambda}_n \hat{w}_n(\hat{\alpha}_n x)(y - x)) \hat{g}_{n1}(\hat{\alpha}_n, x) \\ (29) \quad &=: \exp(-\hat{\lambda}_n \hat{w}_n(\hat{\alpha}_n x)y) \hat{g}_{n1}^*(\hat{\alpha}_n, x),\end{aligned}$$

or alternatively

$$\begin{aligned}\hat{\Psi}_{n,2}(y, x) &:= \frac{\hat{\alpha}_n^{3/2} K_{1,\hat{\rho}_n}}{\sqrt{2\pi}} \exp(-\hat{\lambda}_n \hat{w}_n(\hat{\alpha}_n x)(y - x)) \hat{g}_{n2}(\hat{\alpha}_n, x) \\ (30) \quad &=: \exp(-\hat{\lambda}_n \hat{w}_n(\hat{\alpha}_n x)y) \hat{g}_{n2}^*(\hat{\alpha}_n, x).\end{aligned}$$

Let $\Phi(q, x), q \in (0, 1), x > 0$ be the quantile function defined as the inverse function of $1 - \Psi(y, x)$ for $x > 0$ fix. Inverting the above expressions we have also two estimators of the conditional quantile function

$$(31) \quad \hat{y}_{n,1}(q, x) := \frac{\ln(\hat{g}_{n1}^*(\hat{\alpha}_n, x)) - \ln(1 - q)}{\hat{\lambda}_n \hat{w}_n(\hat{\alpha}_n x)},$$

and

$$(32) \quad \hat{y}_{n,2}(q, x) := \frac{\ln(\hat{g}_{n2}^*(\hat{\alpha}_n, x)) - \ln(1 - q)}{\hat{\lambda}_n \hat{w}_n(\hat{\alpha}_n x)}.$$

The difficulty in constructing these estimates lies in the fact that we have to estimate both the tail $1 - F$ (implicit in the estimation of the functions g_1, g_2), and the scaling function w . In the setup of Example 3 above there is a simple relation about these functions, and the second order correction is easy to handle.

6. PROOFS AND RELATED ASYMPTOTICS

In the following lemma we derive a formula for the distribution function of a bivariate elliptical random vector. Define next for $a \geq 1$ and x, y positive constants

$$(33) \quad I(a, x) := \int_a^\infty [1 - F(xs)] \frac{1}{s\sqrt{s^2 - 1}} ds,$$

and

$$(34) \quad \alpha_{\rho,x,y} := \sqrt{1 + ((y/x) - \rho)^2/(1 - \rho^2)} \geq 1, \quad \beta_{\rho,x,y} := \alpha_{\rho,x,y} x/y, \quad x, y \in \mathbb{R}, \rho \in (-1, 1).$$

Lemma 5. *Let (S_1, S_2) be a bivariate spherical random vector with associate random radius R which has distribution function F . If $\rho \in [0, 1]$ and $F(0) = 0$ then we have: a) If $x > 0$ and $y \in (\rho x, x]$*

$$(35) \quad \mathbf{P}\{S_1 > x, \rho S_1 + \sqrt{1 - \rho^2} S_2 > y\} = \frac{1}{2\pi} [I(\alpha_{\rho,x,y}, x) + I(\beta_{\rho,x,y}, y)].$$

b) If $y/x < \rho$ and $x > 0, y \geq 0$

$$(36) \quad \mathbf{P}\{S_1 > x, \rho S_1 + \sqrt{1-\rho^2} S_2 > y\} = \frac{1}{2\pi} [2I(1, x) - I(\alpha_{\rho, x, y}, x) + I(\beta_{\rho, x, y}, y)].$$

Proof. We have the stochastic representation (see Cambanis et al. (1981), Berman (1992))

$$\begin{aligned} (S_1, \rho S_1 + \sqrt{1-\rho^2} S_2) &= (S_2, \rho S_2 + \sqrt{1-\rho^2} S_1) \\ &\stackrel{d}{=} (S_1 \cos(\Theta), S_2 \cos(\Theta - \psi)), \end{aligned}$$

with Θ uniformly distributed in $(-\pi, \pi)$ independent of R and $\psi := \arccos(\rho)$. For $x > 0, y \geq 0$ two constants we may thus write (see Lemma 3.3 in Hashorva (2005b))

$$\begin{aligned} 2\pi \mathbf{P}\{R \cos(\Theta) > x, R \cos(\Theta - \psi) > y\} \\ &= \int_{\arctan((y/x-\rho)/\sqrt{1-\rho^2})}^{\pi/2} \mathbf{P}\{R > x/\cos(\theta)\} d\theta + \int_{\arccos(\rho)-\pi/2}^{\arctan((y/x-\rho)/\sqrt{1-\rho^2})} \mathbf{P}\{R > y/\cos(\theta - \psi)\} d\theta \\ &= \int_{\arctan((y/x-\rho)/\sqrt{1-\rho^2})}^{\pi/2} \mathbf{P}\{R > x/\cos(\theta)\} d\theta + \int_{\arctan((y/x-\rho)/\sqrt{1-\rho^2})}^{\pi/2-\psi} \mathbf{P}\{R > y/\cos(\theta + \psi)\} d\theta. \end{aligned}$$

As in Abdous et al. (2007) we obtain for $y/x \geq \rho$

$$2\pi \mathbf{P}\{R \cos(\Theta) > x, R \cos(\Theta - \psi) > y\} = I(\alpha_{\rho, x, y}, x) + I(\beta_{\rho, x, y}, y),$$

and if $y/x < \rho$ with x, y positive

$$2\pi \mathbf{P}\{R \cos(\Theta) > x, R \cos(\Theta - \psi) > y\} = 2I(1, x) - I(\alpha_{\rho, x, y}, x) + I(\beta_{\rho, x, y}, y),$$

with $\alpha_{\rho, x, y}, \beta_{\rho, x, y}$ as defined in (34), hence the proof is complete. \square

Note in passing that

$$(S_1 \cos(\Theta), S_2 \cos(\Theta - \arccos(\rho))) \stackrel{d}{=} (S_1 \cos(\Theta), S_2 \sin(\Theta + \arcsin(\rho))),$$

which leads to the alternative formula derived in Abdous et al. (2007) and Klüppelberg et al. (2007) for the tails of elliptical distributions. Remark further that some alternative formulae for the distribution of bivariate elliptical random are presented in Lemma 3.3 in Hashorva (2005b).

In the next lemma we consider a real function $a(x) > 1, \forall x \in \mathbb{R}$. We write for notational simplicity a instead of $a(x)$.

Lemma 6. *Let F satisfy (3) with the scaling function w and $x_F = \infty, F(0) = 0$. Assume further that the Assumption A1 holds.*

i) If $\int_0^\infty B(s) ds < \infty$, then for any function $a := a(x) > 1, x \in \mathbb{R}$ and x large we have

$$(37) \quad \int_a^\infty [1 - F(xs)] \frac{1}{s\sqrt{s^2 - 1}} ds = \frac{1}{a\sqrt{a^2 - 1}} \frac{1 - F(ax)}{xw(ax)} \left[1 + O\left(A(ax) + \frac{1}{xw(ax)} \right) \right].$$

ii) If $z_x, x \in \mathbb{R}$ is such that for all x large $0 \leq z_x < K < \infty$ and further $\int_{z_x}^\infty B(s)/\sqrt{s} ds < \infty$, then for all large x we have

$$(38) \quad \int_{1+z_x/(xw(x))}^\infty [1 - F(xs)] \frac{1}{s\sqrt{s^2 - 1}} ds = \frac{1 - F(x)}{\sqrt{xw(x)}} \sqrt{2\pi} [1 - \Phi(\sqrt{2z_x})] \left[1 + O\left(A(x) + \frac{1}{xw(x)} \right) \right].$$

Proof. i) Let x be a given positive constant. Set

$$a := a(s), \quad v_a(s) := sw(as), \quad s \geq 0$$

and define $I(a, x), \alpha_{\rho, x, y}, \beta_{\rho, x, y}$ as in (33) and (34), respectively. Transforming the variables we have

$$I(a, x) = \frac{1 - F(ax)}{v_a(x)} \int_0^\infty \frac{1 - F(ax + s/w(as))}{1 - F(ax)} \frac{1}{(a + s/v_a(x))\sqrt{(a + s/v_a(x))^2 - 1}} ds.$$

Further (9) implies for any $s \geq 0$ and all x large

$$\left| \frac{1 - F(ax + s/w(as))}{1 - F(ax)} - \exp(-s) \right| \leq A(ax)B(s).$$

Hence utilising further the Assumption A1 we may write for all x large

$$\begin{aligned}
& \left| \frac{v_a(x)}{1 - F(ax)} \int_a^\infty [1 - F(xs)] \frac{1}{s\sqrt{s^2 - 1}} ds - \int_0^\infty \frac{\exp(-s)}{a\sqrt{a^2 - 1}} ds \right| \\
& \leq \int_0^\infty \left| \frac{1 - F(ax + s/w(ax))}{1 - F(ax)} - \exp(-s) \right| \frac{1}{(a + s/v_a(x))\sqrt{(a + s/v_a(x))^2 - 1}} ds \\
& \quad + \int_0^\infty \exp(-s) \left| \frac{1}{(a + s/v_a(x))\sqrt{(a + s/v_a(x))^2 - 1}} - \frac{1}{a\sqrt{a^2 - 1}} \right| ds \\
& \leq \frac{1}{a\sqrt{a^2 - 1}} A(ax) \int_0^\infty B(s) ds + O\left(\frac{1}{v_a(x)}\right) \\
& = O(A(ax) + \frac{1}{v_a(x)}),
\end{aligned}$$

thus the first claim follows.

ii) Next we consider the second case $a(x) := 1 + z_x/(xw(x))$, $x > 0$. Set $h(x) := xw(x)$, $x > 0$. Transforming the variables we obtain for all x, y positive

$$I(1 + z_x/h(x), x) = \frac{1 - F(x)}{\sqrt{h(x)}} \int_{z_x}^\infty \frac{1 - F(x + s/w(x))}{1 - F(x)} \frac{1}{\sqrt{h(x)(1 + s/h(x))\sqrt{(1 + s/h(x))^2 - 1}}} ds.$$

Hence since $\int_{z_x}^\infty s^{-1/2} B(s) ds < \infty$ for all x large we have (recall Assumption A1)

$$\begin{aligned}
& \left| \int_{z_x}^\infty \frac{1 - F(x + s/w(x))}{1 - F(x)} \frac{1}{\sqrt{h(x)(1 + s/h(x))\sqrt{(1 + s/h(x))^2 - 1}}} ds - \int_{z_x}^\infty \exp(-s) \frac{1}{\sqrt{2s}} ds \right| \\
& \leq \int_{z_x}^\infty \left| \frac{1 - F(x + s/w(x))}{1 - F(x)} - \exp(-s) \right| \frac{1}{\sqrt{h(x)(1 + s/h(x))\sqrt{(1 + s/h(x))^2 - 1}}} ds \\
& \quad + \int_{z_x}^\infty \exp(-s) \left| \frac{1}{\sqrt{h(x)(1 + s/h(x))\sqrt{(1 + s/h(x))^2 - 1}}} - \frac{1}{\sqrt{2s}} \right| ds \\
& \leq A(x) \int_{z_x}^\infty B(s) \frac{1}{\sqrt{2s}} ds + O\left(\frac{1}{h(x)}\right) \int_0^\infty \exp(-s) \sqrt{s} ds \\
& = O(A(x) + \frac{1}{h(x)}).
\end{aligned}$$

Consequently since z_x is bounded for all x large

$$\begin{aligned}
I(1 + z_x/h(x), x) & = \frac{1 - F(x)}{\sqrt{h(x)}} \int_{z_x}^\infty \exp(-s) \frac{1}{\sqrt{2s}} ds \left[1 + O(A(x) + \frac{1}{h(x)}) \right] \\
& = \frac{1 - F(x)}{\sqrt{h(x)}} \sqrt{2\pi} [1 - \Phi(\sqrt{2z_x})] \left[1 + O(A(x) + \frac{1}{h(x)}) \right]
\end{aligned}$$

is valid with Φ the standard Gaussian distribution function on \mathbb{R} . Thus the proof is complete. \square

PROOF OF THEOREM 1 Define $I(a, x)$ and $\alpha_{\rho, x, y}, \beta_{\rho, x, y}$ as in (33) and (34), respectively. Assumption A1 implies that for any $x > 0, \varepsilon > 0$ and u large we have

$$\begin{aligned}
& \left| \frac{1 - F(u + x/w(u))}{1 - F(u)} - \exp(-x) \right| \\
& \leq \frac{d(u + x/w(u))}{d(u)} \left| \frac{1 - F^*(u + x/w(u))}{1 - F^*(u)} - \exp(-x) \right| + \exp(-x) \left| \frac{d(u + x/w(u))}{d(u)} - 1 \right| \\
& \leq (1 + \varepsilon) A_1(u) B_1(x) + \exp(-x) A_2(u) B_2(x),
\end{aligned}$$

with F^* the von Mises distribution function given in (5). We assume for simplicity in the following that the function $d(\cdot)$ is a constant for all x large, implying $A_2(u) = 0$ for all u large.

a) Let x, y be two positive constants such that $y > \rho x$ and $\alpha_{\rho,x,y} \geq c > 1$. In order to complete the proof we need a formula for the survival probability $\mathbf{P}\{X > x, Y > y\}$. In view of (35) we have for $y > \rho x$ and x, y positive

$$2\pi\mathbf{P}\{X > x, Y > y\} = I(\alpha_{\rho,x,y}, x) + I(\beta_{\rho,x,y}, y).$$

Since further $\alpha_{\rho,x,y} \geq c > 1$, applying Lemma 6 we obtain

$$\begin{aligned} 2\pi\mathbf{P}\{X > x, Y > y\} &= \frac{1}{\alpha_{\rho,x,y}\sqrt{\alpha_{\rho,x,y}^2 - 1}} \frac{1 - F(\alpha_{\rho,x,y}x)}{xw(\alpha_{\rho,x,y}x)} \left[1 + O\left(A(\alpha_{\rho,x,y}x) + \frac{1}{xw(\alpha_{\rho,x,y}x)}\right) \right] \\ &\quad + \frac{1}{\alpha_{\rho,x,y}(x/y)\sqrt{\alpha_{\rho,x,y}^2(x/y)^2 - 1}} \frac{1 - F(\alpha_{\rho,x,y}x)}{yw(\alpha_{\rho,x,y}x)} \left[1 + O\left(A(\alpha_{\rho,x,y}x) + \frac{1}{xw(\alpha_{\rho,x,y}x)}\right) \right] \\ &= \frac{1 - F(\alpha_{\rho,x,y}x)}{xw(\alpha_{\rho,x,y}x)} \left[\frac{1}{\alpha_{\rho,x,y}\sqrt{\alpha_{\rho,x,y}^2 - 1}} + \frac{1}{\alpha_{\rho,x,y}\sqrt{\alpha_{\rho,x,y}^2(x/y)^2 - 1}} + O\left(A(\alpha_{\rho,x,y}x) + \frac{1}{xw(\alpha_{\rho,x,y}x)}\right) \right] \\ &= \frac{1 - F(\alpha_{\rho,x,y}x)}{xw(\alpha_{\rho,x,y}x)} \left[\frac{x^2(1 - \rho^2)^{3/2}\alpha_{\rho,x,y}}{(y - \rho x)(x - \rho y)} + O\left(A(\alpha_{\rho,x,y}x) + \frac{1}{xw(\alpha_{\rho,x,y}x)}\right) \right]. \end{aligned}$$

b) Let $z_x, x \in \mathbb{R}$ be constants bounded for all x . We may write for all x, y

$$\mathbf{P}\{Y > y|X > x\} = \mathbf{P}\{X > x\} \frac{\mathbf{P}\{X > x, Y > y\}}{\mathbf{P}\{X > x\}} =: \mathbf{P}\{X > x\}\chi(x, y).$$

For any x positive

$$\mathbf{P}\{X > x\} = \frac{1}{\pi} I(1, x),$$

hence Lemma 6 implies for all x large enough

$$\mathbf{P}\{X > x\} = \frac{1}{\sqrt{2\pi}} \frac{1 - F(x)}{\sqrt{h(x)}} \left[1 + O\left(A(x) + \frac{1}{h(x)}\right) \right],$$

with $h(x) := xw(x), x > 0$. In view of Theorem 3 in Abdous et al. (2007) we have for all large x

$$\chi(x, y) = [1 - \Phi(z_x)] \left[1 + O\left(A(x) + \frac{1}{h(x)}\right) \right].$$

Hence for all large x

$$\mathbf{P}\{X > x, Y > y\} = \frac{1}{\sqrt{2\pi}} \frac{1 - F(x)}{\sqrt{h(x)}} [1 - \Phi(z_x)] \left[1 + O\left(A(x) + \frac{1}{h(x)}\right) \right]^2,$$

thus the result follows.

c) Now we consider the last case. By (36) for all x, y large $y \leq ax < \rho x$ (implying $y \in (0, x]$)

$$2\pi\mathbf{P}\{X > x, Y > y\} = 2I(1, x) - I(\alpha_{\rho,x,y}, x) + I(\beta_{\rho,x,y}, y),$$

where $\alpha_{\rho,x,y} \geq \sqrt{1 + (a - \rho)^2/(1 - \rho)^2} > 1$ and $\beta_{\rho,x,y} := \alpha_{\rho,x,y}x/y > 1$. By the above results we have for all $y < ax$ and x, y large enough

$$\mathbf{P}\{Y > y|X > x\} = \frac{1}{\sqrt{2\pi}} \frac{1 - F(x)}{\sqrt{h(x)}} \left[1 + O\left(A(x) + \frac{1}{h(x)} + \frac{1}{\sqrt{h(x)}} \frac{1 - F(\alpha_{\rho,x,y}x)}{1 - F(x)}\right) \right],$$

thus the proof is complete. \square

PROOF OF THEOREM 2 From the proof of Theorem 1 we obtain for all large x

$$\mathbf{P}\{X > x\} = \frac{1 - F(x)}{\sqrt{2\pi x w(x)}} \left[1 + O\left(A(x) + \frac{1}{xw(x)}\right) \right].$$

Since further $\alpha_{\rho,x,y} \geq 1$ and it is bounded for all x large and y positive such that (12) holds, we may write for all x large

$$(39) \quad \mathbf{P}\{X > \alpha_{\rho,x,y}x\} = \frac{1 - F(\alpha_{\rho,x,y}x)}{\sqrt{2\pi\alpha_{\rho,x,y}xw(\alpha_{\rho,x,y}x)}} \left[1 + O\left(A(\alpha_{\rho,x,y}x) + \frac{1}{xw(\alpha_{\rho,x,y}x)}\right) \right].$$

Applying Theorem 1 we have

$$\begin{aligned} & \mathbf{P}\{X > x, Y > y\} \\ &= \frac{\alpha_{\rho,x,y}^{3/2} K_{\rho,x,y}}{\sqrt{2\pi xw(\alpha_{\rho,x,y}x)\pi}} \frac{1 - F(\alpha_{\rho,x,y}x)}{\sqrt{2\pi\alpha_{\rho,x,y}xw(\alpha_{\rho,x,y}x)}} \mathbf{P}\{X > \alpha_{\rho,x,y}x\} \\ & \quad \times \left[1 + O\left(A(\alpha_{\rho,x,y}x) + \frac{1}{xw(\alpha_{\rho,x,y}x)}\right) \right] \\ &= \frac{\alpha_{\rho,x,y}^{3/2} K_{\rho,x,y}}{\sqrt{2\pi xw(\alpha_{\rho,x,y}x)}} \mathbf{P}\{X > \alpha_{\rho,x,y}x\} \left[1 + O\left(A(\alpha_{\rho,x,y}x) + \frac{1}{xw(\alpha_{\rho,x,y}x)}\right) \right]^2 \\ &= \frac{\alpha_{\rho,x,y}^{3/2} K_{\rho,x,y}}{\sqrt{2\pi xw(\alpha_{\rho,x,y}x)}} \mathbf{P}\{X > \alpha_{\rho,x,y}x\} \left[1 + O\left(A(\alpha_{\rho,x,y}x) + \frac{1}{xw(\alpha_{\rho,x,y}x)}\right) \right]. \end{aligned}$$

Thus the result follows. \square

PROOF OF THEOREM 3 The proof of the first claim follows easily since

$$\mathbf{P}\{Y > x | X > x\} = \mathbf{P}\{X > x, Y > y\} / \mathbf{P}\{X > x\}, \quad \forall x > 0$$

utilising further the results of Theorem 1 and (39). The fact that the distribution function F is in the Gumbel max-domain of attraction with scaling function w implies that also the random variable X has distribution function in the Gumbel max-domain of attraction (Berman (1992), Hashorva (2005b)) with the same scaling function w . If y are positive constants such that $\alpha_{\rho,x,y} > c > 1$ holds, then for all large x

$$\lim_{x \rightarrow \infty} \frac{\mathbf{P}\{X > \alpha_{\rho,x,y}x\}}{\mathbf{P}\{X > x\}} = 0,$$

hence the result follows. \square

PROOF OF LEMMA 4 Since $F_i, i \geq 1$ are in the Gumbel max-domain of attraction with the same scaling function w if follows easily that the distribution function F is in the Gumbel max-domain of attraction with the same scaling function w . It can be easily checked that both Theorem 2 and Theorem 3 hold with $A(u) := \sum_{i=1}^{\infty} A_{ii}(u)$, $u > 0$, hence the result follows. \square

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